Mathematical Modeling Seminar

Modern Portfolio Theory

& Capital Asset Pricing Model

(Introduction)

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BMETE15MF78
github.com/fij/btup

About Citi

CEO	Jane Fraser	220 000+ colleagues globally
Managing Director, Citi Country Officer, and Banking Head for Hungary	Veronika Spanarova	3 000+ in Budapest

- Accounts
- Online Banking
- Trade Solutions
- Finance & Lending
- Custody
- Treasury

About myself

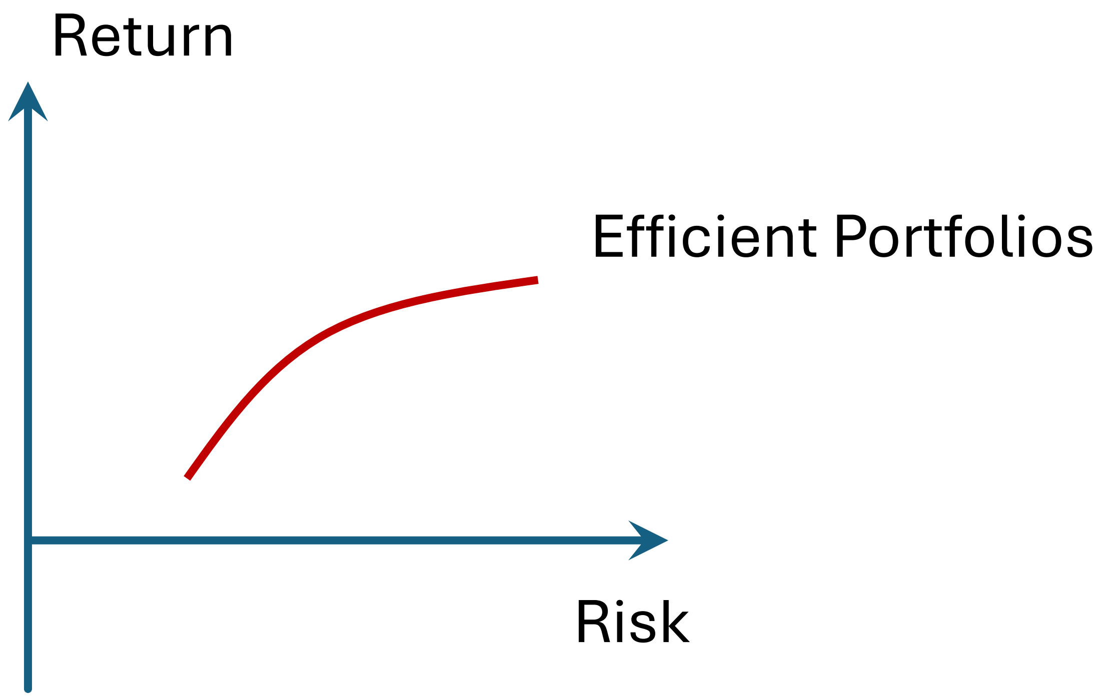
2004	Physics PhD at Eotvos University, Advisor: Tamas Vicsek
– 2017	Eotvos Univ. / Hung. Acad. of Sci. Habilitation, DSc
2017 –	Citi HU MQA (Markets Quantitative Analysis)

Agenda

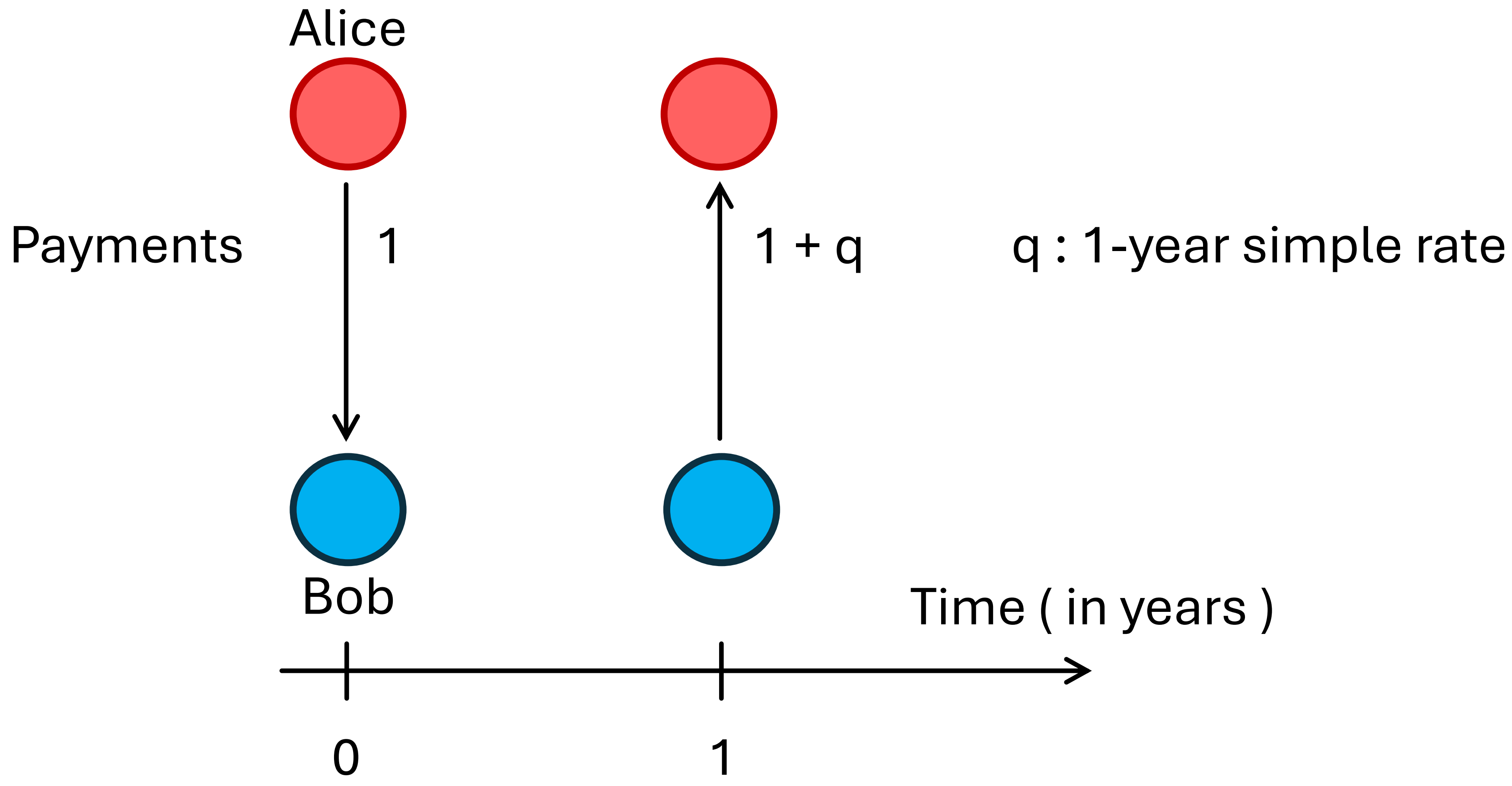
A brief introduction to return, risk, and portfolios

Modern Portfolio Theory (MPT)

Capital Asset Pricing Model (CAPM)



Return



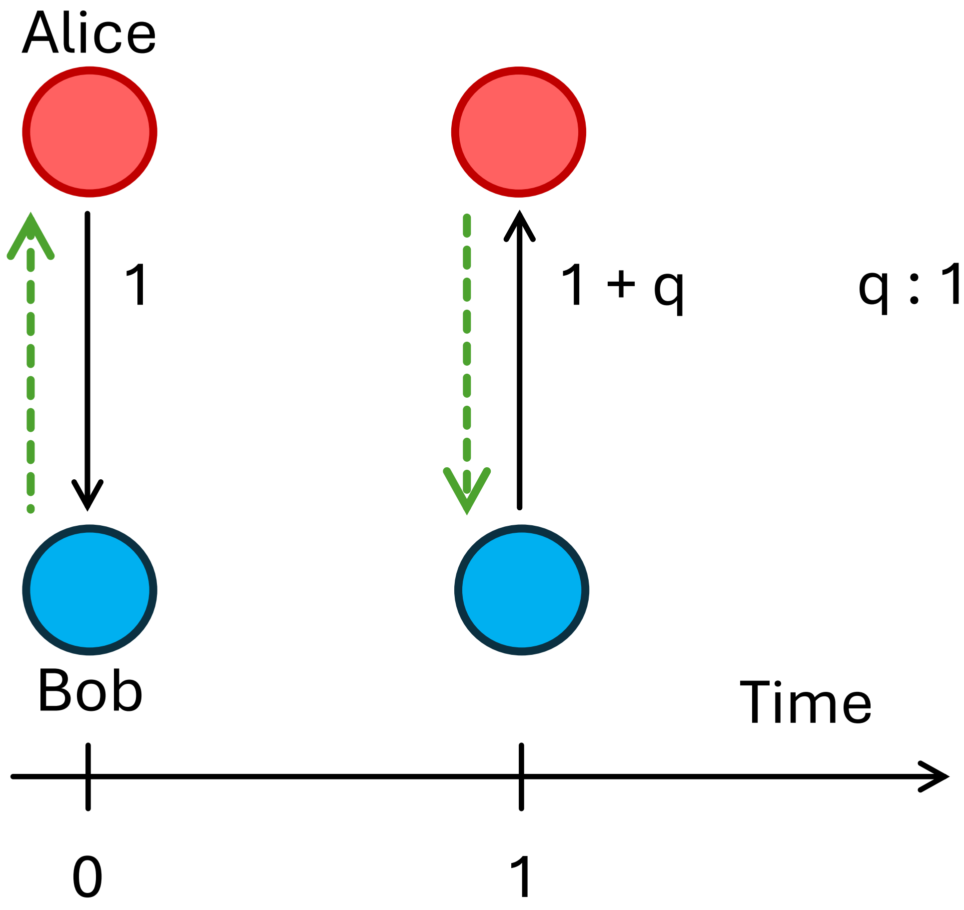
Introduction

Asset

something that has value

Payments

Usually at both payments an asset is transferred in the opposite direction



q : 1-year simple rate

Portfolio

A linear combination of assets

For example, 1 unit of MSFT and 1 unit of AAPL

Normalization to sum of weights = 1 gives :   0.5 MSFT + 0.5 AAPL

Introduction

Risk

(for a single asset)

Any contract contains numbers that will be known only in the future

Risk quantifies the **uncertainty** of a **future** value, for example, the future price of an asset

We don't know the future, but we can **forecast**

A simple forecast for the uncertainty of an asset's future price is the same asset's **past** **volatility**

Why forecast the return ?

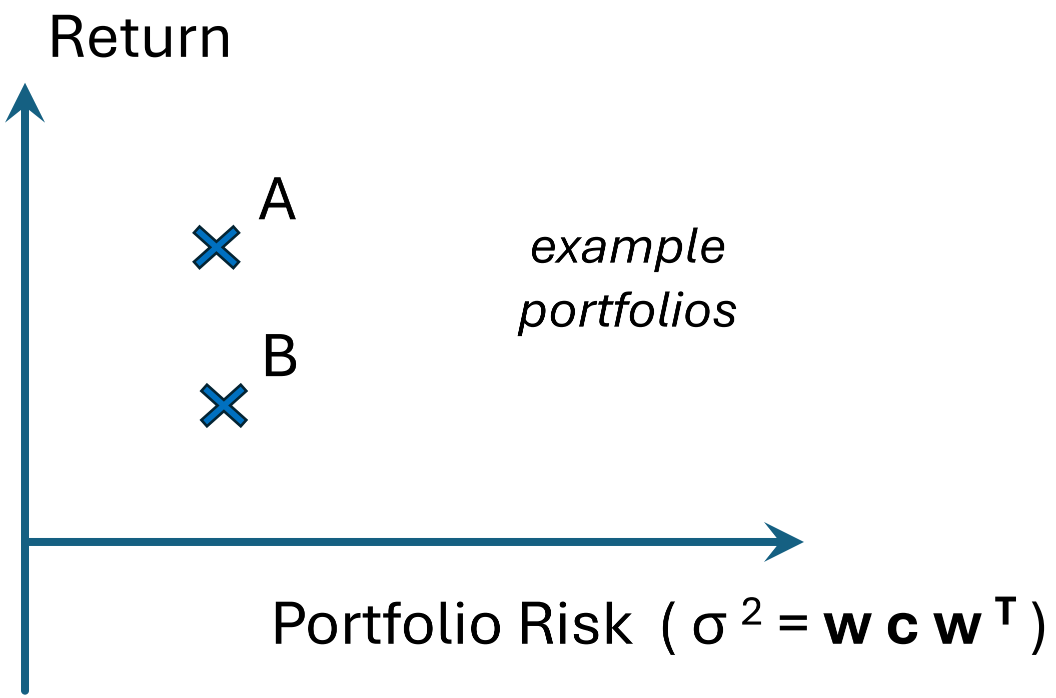
q can be used for discounting

discounting can be applied to – for example – bond pricing

Modern Portfolio Theory (MPT)

N **risky** assets :

Variable weights w_i	$\mathbf{w}$	row vectors
Fixed yearly returns r_i	$\mathbf{r}$	
and covariance c_{ij}	$\mathbf{c}$	matrix



What is the max q ?

(q does have a max, for example, $q \leq \max r_i$)

Equivalently : What is the min σ^2 ?

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What is the min σ^2 ?

Fixed: covariance matrix and return. Variable: weight vector.

Find min σ^2 with 2 constraints : sum of weights is 1 , portfolio return is q.

Lagrange method

$$\searrow \quad \sigma^2(q) \longrightarrow q(\sigma^2) \xrightarrow[\text{Variable transformation}]{\text{Add risk-free asset}} q(\sigma)$$

Risky assets only

What is the min σ^2 ?

Following the description of the task and the Lagrange method, let's apply two new scalar variables, $-\lambda_1$ and $-\lambda_q$, and `minimize the Lagrange function` $\mathcal{L}\left(\mathbf{w}, \lambda_1, \lambda_q\right) = \mathbf{w}\,\mathbf{c}\,\mathbf{w}^{\mathrm{T}} - \lambda_1\left(\mathbf{w}\,\mathbf{1}^{\mathrm{T}} - 1\right) - \lambda_q\left(\mathbf{w}\,\mathbf{r}^{\mathrm{T}} - q\right)$.

The necessary $\mathbf{0} = \boldsymbol{\nabla}\mathcal{L}$ condition for a $\left(\mathbf{w}, \lambda_1, \lambda_q\right)$ vector to be the location of a local minimum has three parts :

$$(1) \quad \mathbf{0} = \frac{\partial \mathcal{L}}{\partial \mathbf{w}} = \mathbf{c}\,\mathbf{w}^{\mathrm{T}} - \lambda_1\,\mathbf{1}^{\mathrm{T}} - \lambda_q\,\mathbf{r}^{\mathrm{T}}$$

$$(2) \quad 0 = \frac{\partial \mathcal{L}}{\partial \lambda_1} = 1 - \mathbf{w}\,\mathbf{1}^{\mathrm{T}}$$

$$(3) \quad 0 = \frac{\partial \mathcal{L}}{\partial \lambda_q} = q - \mathbf{w}\,\mathbf{r}^{\mathrm{T}}$$

What is the min σ^2 ? Equivalently : max q ?

$$A = \begin{pmatrix} \mathbf{1}\,\mathbf{c}^{-1}\,\mathbf{1}^{T} & \mathbf{r}\,\mathbf{c}^{-1}\,\mathbf{1}^{T} \\ \mathbf{r}\,\mathbf{c}^{-1}\,\mathbf{1}^{T} & \mathbf{r}\,\mathbf{c}^{-1}\,\mathbf{r}^{T} \end{pmatrix}$$

$$\frac{1}{K^2} = \frac{1}{A_{22} - (A_{12})^2 / A_{11}}, \quad q_0 = \frac{A_{12}}{A_{11}}, \quad \sigma_0^2 = \frac{1}{A_{11}}$$

$$\sigma^2(q) = \frac{1}{K^2}(q - q_0)^2 + \sigma_0^2$$

$$q(\sigma^2) = q_0 + K\sqrt{\sigma^2 - \sigma_0^2}$$

Efficient Frontier (EF) of the risky assets only

Numerical issues

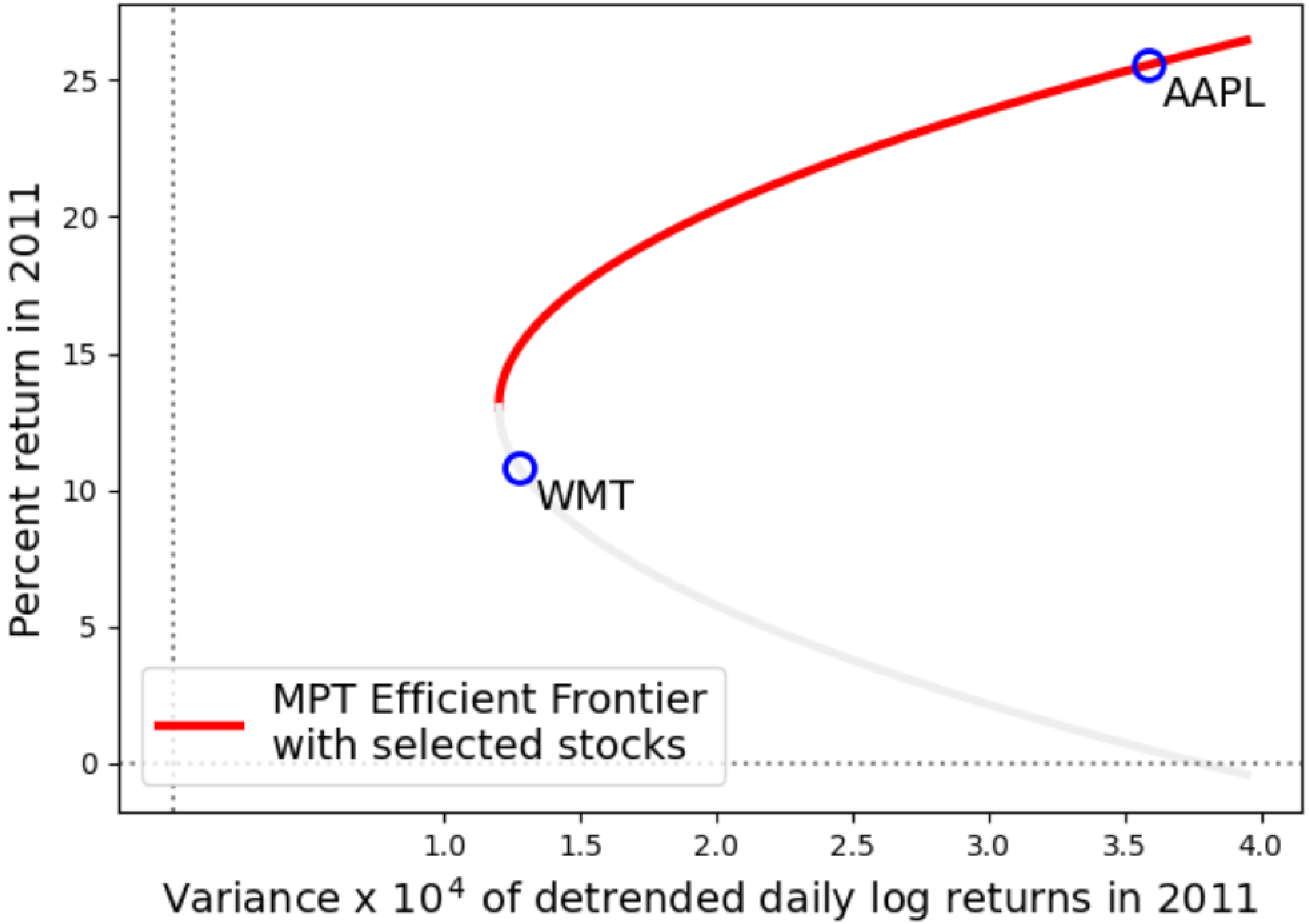
$\mathbf{c}^{-1}$

$\mathbf{A}^{-1}$

Modern Portfolio Theory (MPT)

Example :

Efficient Frontier of 2 risky assets

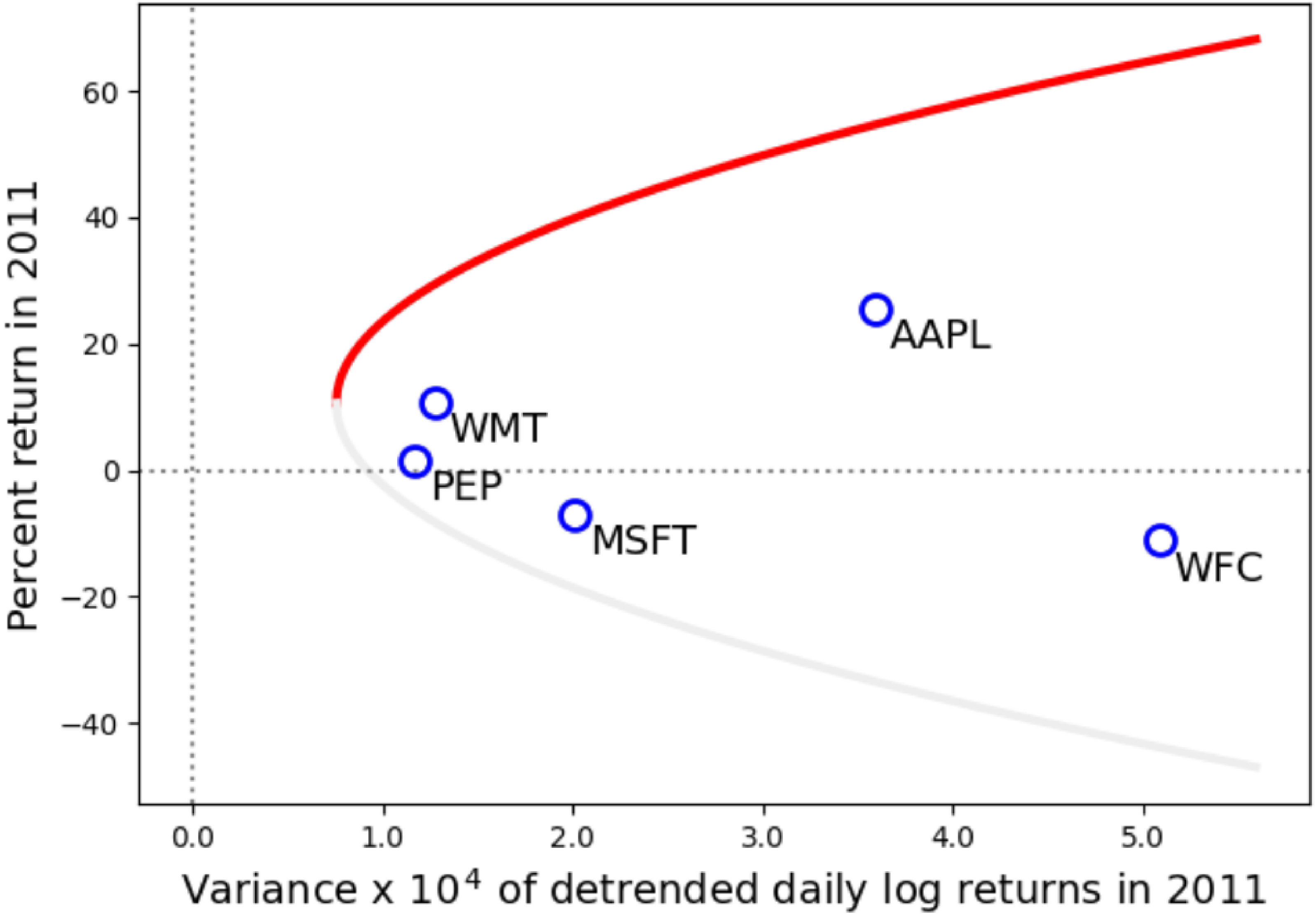


Data source:
kaggle.com/dgawlik/nyse

Modern Portfolio Theory (MPT)

Example :

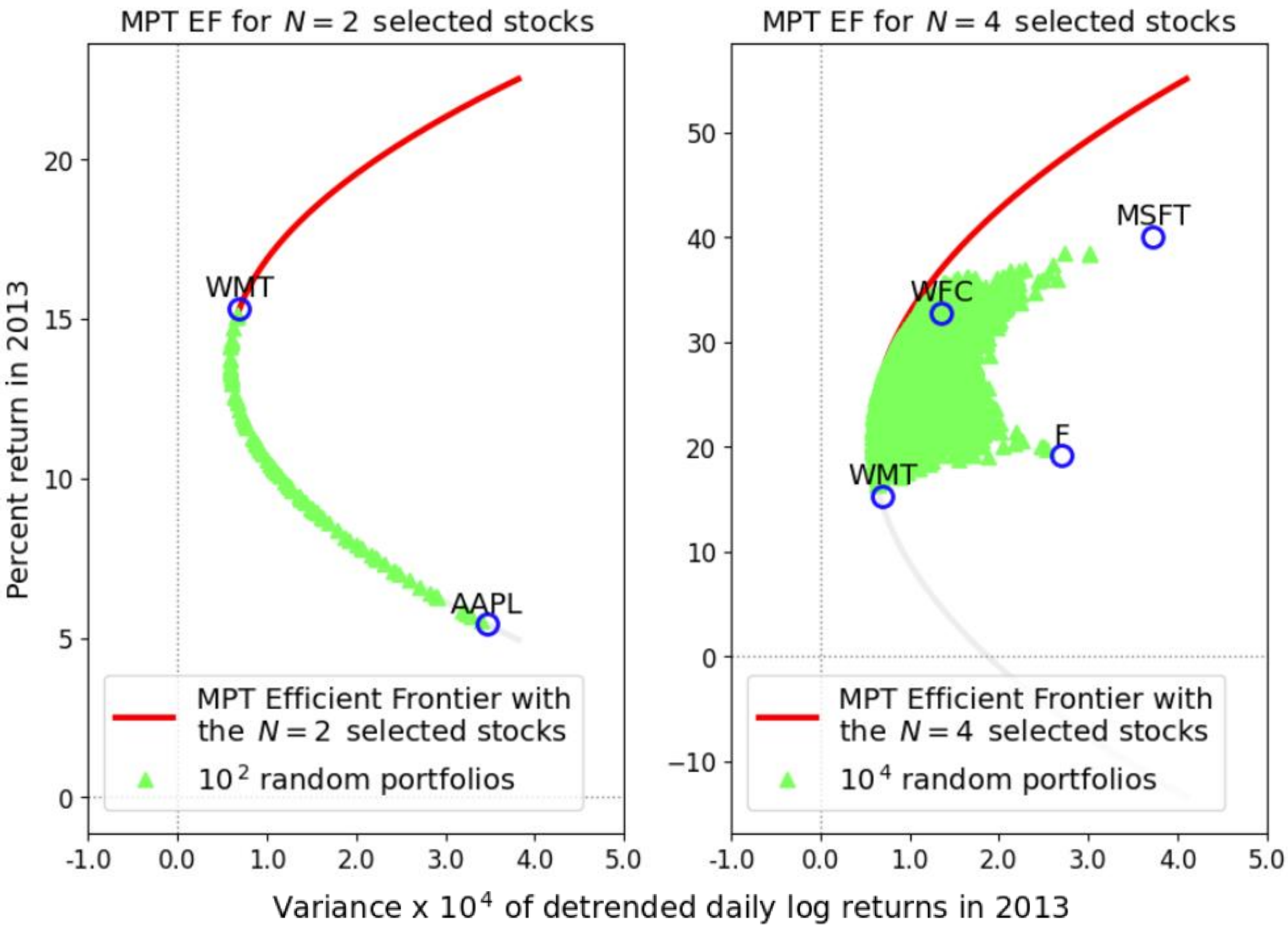
Efficient Frontier of 5 risky assets



Modern Portfolio Theory (MPT)

Example :

Rnd portfolios
of risky assets



Variable change : $\sigma^2 \rightarrow \sigma$

$$q(\sigma) = q_0 + K\sqrt{\sigma^2 - \sigma_0^2} \quad \text{with} \quad \sigma^2 \geq \sigma_0^2$$

$$\frac{\sigma^2}{\sigma_0^2} - \frac{(q - q_0)^2}{K^2 \sigma_0^2} = 1$$

Recall that in the MPT only the $(\sigma > \sigma_0,\ q > q_0)$ quadrant of the hyperbola is used.

From the above standard equation form note that the `center of the hyperbola` is at $(\sigma = 0,\ q = q_0)$.

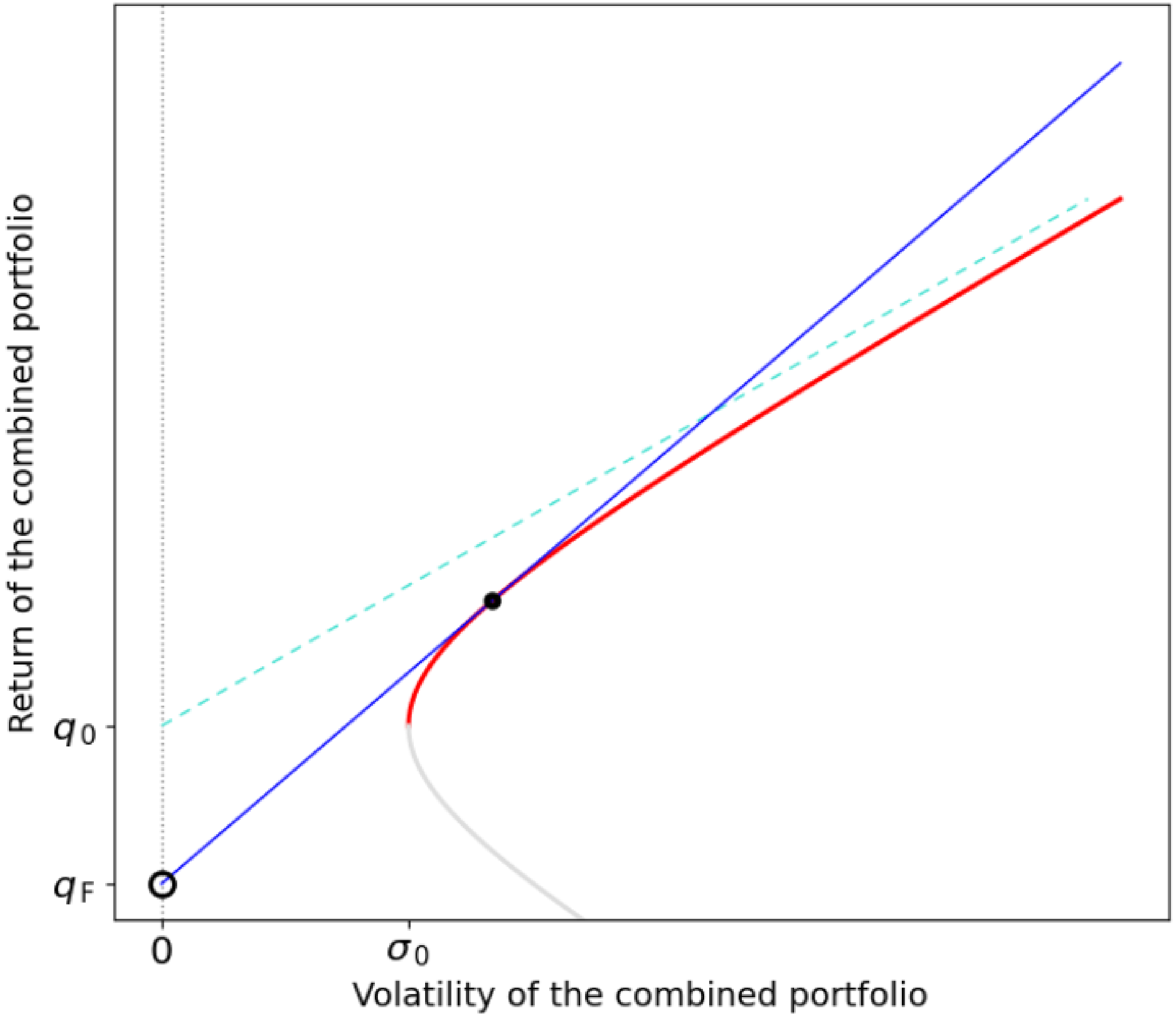
Take the $\sigma \rightarrow \infty$ limit (meaning also $q \rightarrow \infty$) to see that the `asymptote` of the hyperbola is $q = q_0 + K\sigma$.

Modern Portfolio Theory (MPT)

Efficient Frontier (EF) of N risky assets plus 1 risk-free asset :

Line containing the Risk-Free point and tangent to the Risky EF

Illustration of the Efficient Frontier of the combined portfolio of risky assets and the risk-free asset



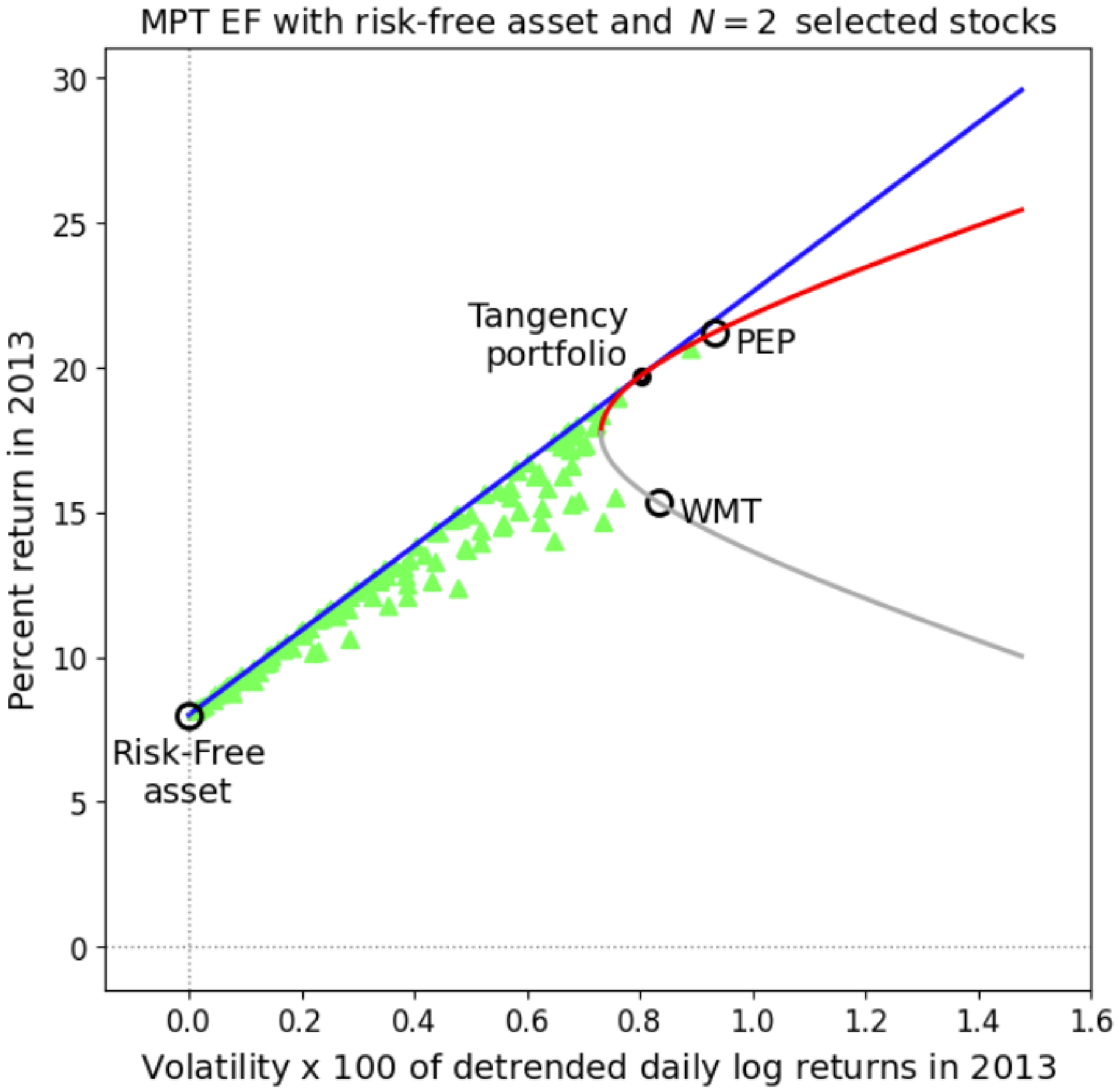
- Efficient Frontier (EF) of the risky assets
- EF of the combined (risky + risk-free) portfolios
- Asymptote ($q = q_0 + K\sigma$) of the EF of risky assets
- Risk-Free asset
- Tangency point (T)

Modern Portfolio Theory (MPT)

Example:

risk-free asset + 2 risky assets

Efficient Frontier
Random portfolios



Capital Asset Pricing Model (CAPM)

MPT	σ	full risk
CAPM	β	non-diversifiable risk

Capital Asset Pricing Model (CAPM)

What does the MPT do ?

`Modern Portfolio Theory` (MPT) uses the Efficient Market Hypothesis (EMH) to calculate
(i) the Efficient Frontier (EF) of portfolios containing only risky assets
(ii) and the Capital Market Line (CML), which is the EF of the combined (risk-free + risky) portfolios.

The EF of risky assets and the CML are tangent at the `tangency point`.

The coordinates of the tangency point are the volatility and return of the `tangency portfolio`.

The tangency portfolio has no risk-free asset. It has relative risky asset weights equal to those of the market portfolio.

The Capital Market Line (CML) compares return with the `full risk`, which is quantified as the volatility.

What does the CAPM do ?

The `Capital Asset Pricing Model` was developed in the 1960s by Jack Treynor and others based on the MPT.

The CAPM compares a risky portfolio's return with the amount of its `non-diversifiable risk` (also called: systematic risk).

Notes

1. This non-diversifiable risk will quantified as the β value defined below.
2. Diversifiable risk is also called specific risk, or unsystematic risk, or idiosyncratic risk.

Capital Asset Pricing Model (CAPM)

β of a portfolio P
(P can be a single asset)

Equivalent definitions
(M is the market portfolio)

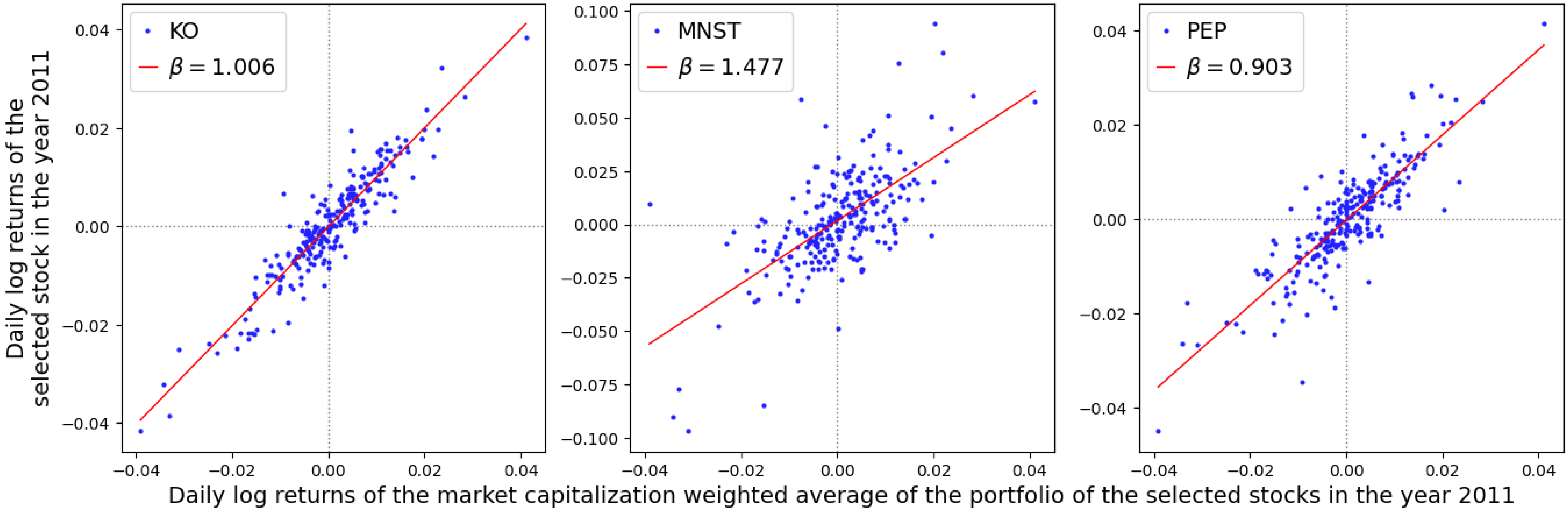
$$\beta_P = \frac{Cov\,(\,P, M\,)}{Var\,(\,M\,)}$$

β_P is the slope of the linear fit to the scatter plot displaying P (vertical coordinate) vs M (horizontal)

β_P is the non-diversifiable (also called: systematic) risk of the portfolio P

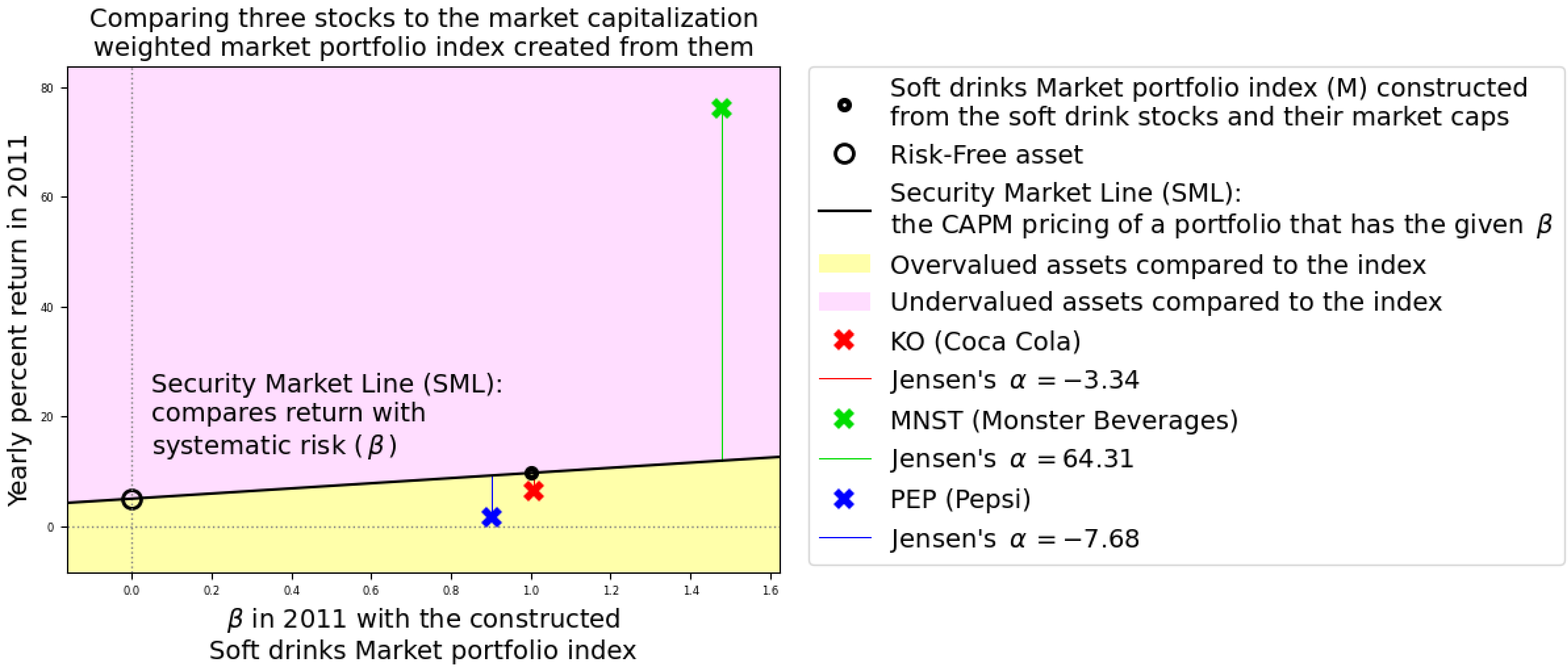
Example: Market is 3 stocks

Pepsi, Coke, Monster Beverages



Capital Asset Pricing Model (CAPM)

Example: Market is 3 stocks



Thank you

Questions